Generic structures of amalgamation classes for irrationals (Model theoretic aspects of the notion of independence and dimension)

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Generic structures of amalgamation classes for irrationals

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Abstract

I talked on the case f by the Hrushovski's construction is bounded in the workshop. But in this report, I will show the other construction of unbounded f for an irrationals.

1 Introduction

Definition 1.1. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and A finite graphs. We define the predimension of A $\delta_\alpha(A) := |A| - \alpha e(A)$ where $e(A)$ is the number of the edges of A.

Let $B \subseteq A$. $\delta_\alpha(A/B) := \delta_\alpha(A) - \delta_\alpha(B)$.

- $B \leq_\alpha A :\iff \delta_\alpha(XA/A) \geq 0$ for all $X \subseteq B$.
- $B <_\alpha A :\iff \delta_\alpha(XA/A) > 0$ for all non empty graph $X \subseteq B \setminus A$. We say that A is *closed* in B.
- $\mathcal{K}_\alpha := \{A : \text{finite graph}, \emptyset <_\alpha A\}$.

Definition 1.2. Let $\mathcal{K}$ be a class of finite graphs closed isomorphism and containing $\emptyset$.

- $\mathcal{K}$ has the *HP (Hereditary Property)* if for all $A \in \mathcal{K}$ and $B \subseteq A$, $B \in \mathcal{K}$.
- $(\mathcal{K}, <_\alpha)$ has the *AP (Amalgamation Property)* if for all $A, B_1, B_2 \in \mathcal{K}$ with $A <_\alpha B_1, B_2$, there is $C \in \mathcal{K}$ and embeddings $g_i : B_i \to C$ such that $g_i(B_i)$ is closed in C and $g_1 \circ f_1 \restriction A = g_2 \circ f_2 \restriction A$.
- $(\mathcal{K}, <_\alpha)$ has the *FAP (Free Amalgamation Property)* if for above f_i's and g_j's, there is no edge between $g_1(B_1) \setminus g_1 \circ f_1(A)$ and $g_2(B_2) \setminus g_2 \circ f_2(A)$.

If $(\mathcal{K}, <_\alpha)$ has the HP and the AP, we call $(\mathcal{K}, <_\alpha)$ an *amalgamation class*. If $(\mathcal{K}, <_\alpha)$ has the HP and the FAP, we call $(\mathcal{K}, <_\alpha)$ a *free amalgamation class*.

Fact 1.3. If $(\mathcal{K}, <_\alpha)$ is the amalgamation class, then there is a countable graph M called a *generic graph* which holds the following conditions:

- For all $A \subseteq_{\mathrm{fin}} M$, there is $B \subseteq_{\mathrm{fin}} M$ such that $A \subseteq B <_\alpha M$.
- Every finite induced subgraph A of M is in $\mathcal{K}$.
- For all $A, B \in \mathcal{K}$ with $A <_\alpha M$ and $A <_\alpha B$, B can be embedded into M.

Fact 1.4. $(\mathcal{K}_\alpha, <_\alpha)$ is free amalgamation class. So $(\mathcal{K}_\alpha, <_\alpha)$ has a generic structure.

Definition 1.5. Let $\mathcal{K}$ be a free amalgamation class. $A \in \mathcal{K}$ is *absolutely closed* if for every $B \in \mathcal{K}$ with $A \subseteq B$, $A <_\alpha B$.

Absolute closedness is concerned with model completeness of M of $(\mathcal{K}, <_\alpha)$.

Proposition 1.6. Let $(\mathcal{K}, <_\alpha)$ be a free amalgamation class and M a generic structure of $(\mathcal{K}, <_\alpha)$. Assume that for every $A \in \mathcal{K}$, there is $C \in \mathcal{K}$ such that $A <_\alpha C$ and C is absolutely closed. Then the theory of M is model complete.

Example 1.7. Let $\alpha = \dfrac{m}{d}$ such that m, d are relatively prime and $f(x) = \dfrac{1}{d}\log_2(x+1)$. Then $\mathcal{K}_{\alpha,f}: = \{A \in \mathcal{K}_\alpha \mid \delta_\alpha(X) > f(|X|) \text{ for all } X \subseteq A\}$ is the free amalgamation class for α and holds the condition in the proposition 1.6, so M has the model complete theory.

The absolutely closedness in the above is due to that f is unbounded. Hrushovski show that there is an unbounded f for uncountably many α's, especially for any rational α. So, we define like some height of f, called *index* of f.

Definition 1.8. Let $\mathcal{K} = \mathcal{K}_{\alpha,f}$ be a free amalgamation class for α.
$\mathrm{ind}_\alpha(\mathcal{K}) := \max_{n<\omega} f(n)$. If f is unbounded, we define $\mathrm{ind}_\alpha(\mathcal{K}) := \infty$.

2 The construction of an unbounded f for an irrational α

Proposition 2.1. Let $0 < \alpha < 1$ If $\mathcal{K}_1, \mathcal{K}_2$ are two free amalgamation classes, then so is $\mathcal{K}_1 \cap \mathcal{K}_2$.

Proof. Obvious. □

Corollary 2.2. Assume that $\mathcal{A}$ is a (finite) class of finite graphs and $\mathfrak{F}_\alpha(\mathcal{A})$ is a class of free amalgamation classes containing $\mathcal{A}$. Then there exists a minimal free amalgamation class $\mathcal{K}$ in $\mathfrak{F}_\alpha(\mathcal{A})$ and $\mathcal{K}$ is the class generated by $\mathcal{A}$ by amalgamating graphs in $\mathcal{A}$.

Note that for $0 < \alpha_1 < \alpha_2 < 1$ and A, B a finite graph, $\delta_{\alpha_1}(A) = |A| - \alpha_1 e(A) \geq |A| - \alpha_2 e(A) = \delta_{\alpha_2}(A)$.

Proposition 2.3. Let $0 < \alpha_1 < \alpha_2 < 1$ and $\mathcal{K}$ be a class of finite graphs. If $\mathcal{K}$ has the FAP for α_1, then it has the FAP for α_2.

Proof. Fix $A, B, C \in \mathcal{K}$ with $A <_{\alpha_2} B, C$. By the above inequality, $A <_{\alpha_1} B, C$. So $B \otimes_A C \in \mathcal{K}$ by the FAP for α_1. Hence $\mathcal{K}$ has the FAP for α_2 □

Lemma 2.4. Let $1 > \alpha > \alpha' = \dfrac{m}{d} > 0$ and $f(x) = \dfrac{1}{d}\log_2(x+1)$.

Then $\mathcal{K}_{\alpha',f} = \mathcal{K}_{\alpha,g}$ where $g(x) = \left(1 - \dfrac{\alpha}{\alpha'}\right)x + \dfrac{\alpha}{m}\log_2(x+1)$.

Proof. Change the variables by $\begin{pmatrix} 1 & 0 \\ 1 - \dfrac{\alpha}{\alpha'} & \dfrac{\alpha}{\alpha'} \end{pmatrix}$. □

Lemma 2.5. Let $1 > \alpha > \alpha' = \dfrac{m}{d} > 0$ and $f(x) = \dfrac{1}{d}\log_2(x+1)$.

Then $\mathrm{ind}_\alpha(\mathcal{K}_{\alpha',f}) = \dfrac{\alpha}{m\log 2} - 1 + \dfrac{\alpha}{\alpha'} + \dfrac{\alpha}{m}\Big\{\log_2\alpha - \log_2 d - \log_2(\alpha - \alpha') - \log_2(\log 2)\Big\}$.

Now we will consider the limit of classes for α_n's obtained in the following. Suppose that $\langle \alpha_n \rangle_{n<\omega} \subseteq \mathbb{Q}_{\geq 0}$ is an increasing sequence converging to α and $\alpha_n = \dfrac{m_n}{d_n}$ such that d_n, m_n are relatively prime. $f_n(x) = \dfrac{1}{d_n}\log_2(x+1)$. We define $\mathcal{K}_n := \bigcap_{i<n} \mathcal{K}_{\alpha_i, f_i}$ and $\mathcal{K} := \lim_{n\to\infty}\mathcal{K}_n$. Then $\mathrm{ind}_\alpha(\mathcal{K}) = -\lim_{n\to\infty}\dfrac{C}{d_n}\log_{10}(\alpha - \alpha_n) + D$ for some $C, D \in \mathbb{R}$. So we may find a sequence α_n with $\alpha - \alpha_n$ rapidly diverges than the speed of d_n, like some Liuville numbers.

Proposition 2.6. *Let* $\alpha_n = \sum_{i<n} \dfrac{1}{10 \uparrow\uparrow (2i+1)}$ *and* $\alpha = \lim_{n\to\infty}\alpha_n$ *where* $x \uparrow\uparrow 0 = 1$ *and* $x \uparrow\uparrow (y+1) = x^{x\uparrow\uparrow y}$. *Then* $\mathcal{K}$ *has the FAP for* α *and* $\mathrm{Ind}_\alpha(\mathcal{K}) = \infty$.

Proof. For all $n < \omega$, $\alpha - \alpha_n < \dfrac{10}{10 \uparrow\uparrow (2n+1)} \ll \dfrac{1}{10\uparrow\uparrow 2n}$. Hence $\log_{10}(\alpha - \alpha_n) \ll \dfrac{1}{10\uparrow\uparrow(2n-1)} = \dfrac{1}{d_n}$. □

Conjecture 2.7. For α above, f constructed by Hrushovski has infinite index.

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